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LETTER TO THE EDITOR

A note on families of *bc*-systems of higher rank

Matthias Schork

FB Mathematik, J W Goethe-Universität, 60054 Frankfurt, Germany

E-mail: schork@math.uni-frankfurt.de

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Abstract

We consider a special family of *bc*-systems of higher rank and discuss some properties of its associated anomaly.

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1. Introduction

The *bc*-system first appeared in bosonic string theory as a gauge fixing ghost system and plays a central role [1], in particular in the path-integral approach to scattering amplitudes (see [2, 3] and the extensive list of references therein). In this approach the final expressions are finite-dimensional integrals of Quillen norms of sections of certain determinant line bundles over the moduli spaces $\mathcal{M}_{g,n}$ (or the compactifications $\overline{\mathcal{M}}_{g,n}$) of n-punctured Riemann surfaces. As there are two contributions to the integrand (one from the string embedding X^μ, $\mu = 1, \dots, d$ and the other from the ghosts $b, c, \bar{b}, \bar{c}$), one may use the famous Mumford formula [4] to trivialise the bundle for special choices of parameters, thus fixing the dimension of space–time to $d = 26$ [5]. Recently, a close cousin of the (chiral) *bc*-system based on vector bundles of higher rank was introduced and some of its properties were studied [6, 7]. Since families of the usual system play such a decisive role in string theory, one should thus consider families of these generalized *bc*-systems too. This is what we start here.

In the following we denote by Σ a Riemann surface of genus $g \geqslant 2$ and by K its canonical bundle, i.e., the holomorphic cotangent bundle. We use the same symbol to denote a holomorphic vector bundle and its associated (locally free) sheaf of germs of sections. We also switch freely between the algebraic and analytic category.

2. The relative *bc*-system and some geometry

Before we begin we briefly recall some geometrical background; for this see, e.g., [3, 8]. Assume $\pi : X \to S$ to be a continous map between varieties and let E be a sheaf on X (e.g., the locally free sheaf of sections of some vector bundle). Then the higher direct image sheaves $R^i\pi_*(E)$ on S are the sheaves associated to the presheaves $U \mapsto H^i(\pi^{-1}(U), E_{|\pi^{-1}(U)})$; loosely speaking, we interprete them as cohomology along the fibre, i.e., $R^i\pi_*(E)_s \simeq H^i(X_s, E_s)$, where $X_s := \pi^{-1}(s)$ and $E_s := E_{|\pi^{-1}(s)}$. The set of coherent sheaves on X

is a semigroup under direct sum and we turn it into a group by factoring out the relation $\mathcal{E} - \mathcal{E}_1 - \mathcal{E}_2$ for every exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$, thus obtaining a free abelian group, the Grothendieck group $K(X)$. Its elements are denoted by $[\mathcal{E}]$ or as formal differences $\mathcal{F} - \mathcal{G}$. Now, let E be a coherent sheaf on X and let π be 'sufficiently nice' (e.g., proper and flat); then the direct images $R^i\pi_*(E)$ on S are coherent too and we can define a map $\pi_! : K(X) \to K(S)$, given by

$$\pi_!([E]) := \sum_{i\geqslant 0}(-1)^i[R^i\pi_*(E)]. \tag{1}$$

We now restrict to the case of a family $\pi : \mathcal{C} \to S$ of projective curves (Riemann surfaces), i.e., the fibers $\Sigma_s := \mathcal{C}_s$ have dimension one; here we imagine that $S \subset \mathcal{M}_g$. In this case (1) reduces to $\pi_!(E) = R^0\pi_*(E) - R^1\pi_*(E)$ since the higher cohomologies vanish. Using that a determinant can be defined for coherent sheaves [9], we may use its multiplicative property to define for elements of $K(S)$: $\det(\mathcal{E} - \mathcal{F}) := \det(\mathcal{E}) \otimes \det(\mathcal{F})^{-1}$. We thus obtain $\det \pi_!(E) = \det(R^0\pi_*(E)) \otimes \det(R^1\pi_*(E))^{-1}$. Let us denote by $\omega := \omega_{\mathcal{C}/S}$ the relative dualizing sheaf (which equals the sheaf of relative one-forms $\Omega^1_{\mathcal{C}/S}$ in smooth points) and by $\omega^\lambda = \omega^{\otimes\lambda}$ its powers for $\lambda \in \mathbb{Z}$. Define $\mathcal{L}_\lambda := \det \pi_!(\omega^\lambda)$; its stalks are given by

$$(\mathcal{L}_\lambda)_s \simeq \det H^0(\Sigma_s, \omega^\lambda_{\Sigma_s}) \otimes (\det H^1(\Sigma_s, \omega^\lambda_{\Sigma_s}))^{-1} \simeq \det(\ker \bar\partial_{\lambda;s}) \otimes (\det(\operatorname{coker} \bar\partial_{\lambda;s}))^{-1}$$

where $\bar\partial_{\lambda;s} : K^\lambda_{\Sigma_s} \to K^\lambda_{\Sigma_s} \otimes \overline{K}_{\Sigma_s}$ is the Dolbeault operator appearing in the action of the (chiral) bc-system of conformal weight $(1 - \lambda, \lambda)$ on Σ_s; the case $\lambda = -1$ is the one appearing in bosonic string theory [1]. Defining naively for each s the virtual vector space $\ker \bar\partial_{\lambda;s} - \operatorname{coker} \bar\partial_{\lambda;s} =: \operatorname{ind} \bar\partial_{\lambda;s}$, we see that $(\mathcal{L}_\lambda)_s \simeq \det \operatorname{ind} \bar\partial_{\lambda;s}$, thus showing the connection to the anomaly of the family $\{\bar\partial_{\lambda;s}\}_{s\in S}$ [10, 11]. Defining the (local) anomaly by $\mathcal{A}_\lambda := c_1(\mathcal{L}_\lambda)$, we may use Grothendieck–Riemann–Roch $\mathrm{Ch}(\pi_!(E)) = \pi_*(\mathrm{Ch}(E)\mathrm{Td}T_{\mathcal{C}/S})$ (where π_* is 'integration along the fibre' and $T_{\mathcal{C}/S} = \omega^{-1}_{\mathcal{C}/S}$ is the relative tangent sheaf) to prove the Mumford formula $\mathcal{L}_\lambda \simeq \mathcal{L}_1^{6\lambda^2-6\lambda+1}$ [4], which we interpret as an anomaly relation:

$$\mathcal{A}_\lambda = (6\lambda^2 - 6\lambda + 1) \cdot \mathcal{A}_1. \tag{2}$$

The anomaly coming from the chiral and antichiral ghost system in the bosonic string is given by $-2\mathcal{A}_{-1} = -26\mathcal{A}_1$, thus forcing $d = 26$ [5]. Note the symmetry of (2) around $\frac{1}{2}$ coming from Serre duality:

$$\mathcal{A}_{1-\lambda} = \mathcal{A}_\lambda. \tag{3}$$

3. The relative *bc*-system of higher rank

A generalized bc-system based on a Hermitian vector bundle E of rank r over a Riemann surface was introduced in [7] (see also [6]). Using the Hodge inner product, the action of this bc_r-system is given by $S[b, c] = \frac{\mathrm{i}}{\pi}\int_\Sigma b \wedge \bar\partial_E c$, where c (resp. b) is a section of E (resp. $K \otimes E^\vee$). Following the approach of Raina [12, 13] for the usual rank-one case, it was shown that the simplest possible case results if we choose E to be stable of degree $d = r(g - 1)$ with $h^0(\Sigma, E) = 0$, i.e., E lies outside the nonabelian theta divisor (this corresponds roughly to choosing an even theta-characteristic α with $\alpha^2 \simeq K$ in the rank-one case). In the case where zero-modes are allowed, one uses appropriate insertions to relate these systems to the one considered before where no zero-modes exist. It turns out that—realizing this idea—a satisfactory treatment (existence and uniqueness of correlation functions) exists for rank r only in degree $d = rs$ with $s = 1, \ldots, g - 2$; here we have assumed without loss of generality $0 \leqslant d < r(g - 1)$. Nevertheless, let us consider as above a family of Riemann surfaces (projective curves) $\pi : \mathcal{C} \to S$ endowed with a family $\mathcal{E} \to \mathcal{C}$ of (stable) vector bundles of rank r and degree d (that is, each restriction $\mathcal{E}_s := \mathcal{E}_{|\Sigma_s}$ is of this type). The zero-modes of the field c (resp. b) on Σ_s are given by $H^0(\Sigma_s, \mathcal{E}_s)$ (resp. $H^0(\Sigma_s, K_{\Sigma_s} \otimes \mathcal{E}_s^\vee) \simeq H^1(\Sigma_s, \mathcal{E}_s)^*$) and we obtain in complete analogy to above

$$\det(\mathrm{ind}\ \bar\partial_{\mathcal{E}_s}) \simeq \det H^0(\Sigma_s, \mathcal{E}_s) \otimes (\det H^1(\Sigma_s, \mathcal{E}_s))^{-1} = (\det \pi_!(\mathcal{E}))_s$$

so that we want to consider $\det \pi_!(\mathcal{E})$ ($\equiv \lambda_{\mathcal{E}}$ in the notation of [14] and $(\mathrm{DET}\ \bar\partial_{\mathcal{E}})^{-1}$ in [15]) on S. The anomaly is in this case defined as $\mathcal{A}_{\mathcal{E}} := c_1(\det \pi_!(\mathcal{E}))$ and we want to determine it with the help of Grothendieck–Riemann–Roch. Therefore, we have to calculate the degree-four part of $\mathrm{Ch}(\mathcal{E})\mathrm{Td}(T_{\mathcal{C}/S})$, which is in general given by

$$\frac{r}{12}c_1^2(\omega) - \frac{1}{2}c_1(\mathcal{E})c_1(\omega) + \frac{1}{2}c_1^2(\mathcal{E}) - c_2(\mathcal{E}). \tag{4}$$

Now, applying π_* and using the definition of the Hodge class $\mathcal{A}_1 = \frac{1}{12}\pi_*(c_1^2(\omega))$ we obtain:

Proposition 1. *Let $\pi : \mathcal{C} \to S$ be a family of projective curves (Riemann surfaces) and $\omega_{\mathcal{C}/S} = \omega$ the relative dualizing sheaf. If $\mathcal{E} \to \mathcal{C}$ is a family of (stable) vector bundles of rank r then the anomaly of the family of associated bc_r-systems is given by*

$$\mathcal{A}_{\mathcal{E}} = r \cdot \mathcal{A}_1 - \frac{1}{2}\pi_*(c_1(\omega)c_1(\mathcal{E})) + \frac{1}{2}\pi_*(c_1^2(\mathcal{E})) - \pi_*(c_2(\mathcal{E})). \tag{5}$$

Since this seems to be all one can say in the general case, we now specialize to a situation where we have more contact to the usual bc-system. We thus assume that $\mathcal{E} = \mathcal{F} \otimes \omega^\lambda$ with $\lambda \in \mathbb{Z}$, i.e., we have a family of '$\mathcal{F}$-valued fields of spin λ'. Here we assume that $\mathcal{F}$ is of a somehow simpler type than the general $\mathcal{E}$. Note that $\mathrm{rank}(\mathcal{E}) = \mathrm{rank}(\mathcal{F}) = r$ and $\deg(\mathcal{E}) = \deg(\mathcal{F}) + 2\lambda r(g-1)$. The anomaly is given as

$$\mathcal{A}_{\mathcal{F},\lambda} := \mathcal{A}_{\mathcal{F}\otimes\omega^\lambda} = c_1(\det \pi_!(\mathcal{F} \otimes \omega^\lambda)).$$

Since the expansion of the Chern class gives $c_1(\mathcal{F} \otimes \omega^\lambda) = r\lambda c_1(\omega) + c_1(\mathcal{F})$, we obtain from (4) the degree-four part

$$\frac{r}{12}(6r\lambda^2 - 6\lambda + 1)c_1^2(\omega) + \frac{1}{2}c_1^2(\mathcal{F}) + (r\lambda - \frac{1}{2})c_1(\omega)c_1(\mathcal{F}) - c_2(\mathcal{F} \otimes \omega^\lambda).$$

Using

$$c_2(\mathcal{F} \otimes \omega^\lambda) = \frac{r(r-1)\lambda^2}{2}c_1^2(\omega) + \lambda(r-1)c_1(\omega)c_1(\mathcal{F}) + c_2(\mathcal{F})$$

this equals

$$\frac{r}{12}(6\lambda^2 - 6\lambda + 1)c_1^2(\omega) + \frac{1}{2}c_1^2(\mathcal{F}) + \frac{2\lambda - 1}{2}c_1(\omega)c_1(\mathcal{F}) - c_2(\mathcal{F}).$$

Again, applying π_* and using the definition of the Hodge class we obtain:

Proposition 2. *Assume that $\mathcal{E} = \mathcal{F} \otimes \omega^\lambda$. The associated anomaly is given by*

$$\mathcal{A}_{\mathcal{F},\lambda} = r \cdot \mathcal{A}_\lambda + \frac{1}{2}\pi_*(c_1^2(\mathcal{F})) + \frac{2\lambda - 1}{2}\pi_*(c_1(\omega)c_1(\mathcal{F})) - \pi_*(c_2(\mathcal{F})). \tag{6}$$

Note that (6) reduces to the usual Mumford formula (2) in case that $\mathcal{F}$ is the trivial bundle of rank one. Choosing for $\mathcal{F}$ the trivial bundle of rank r (which is not stable), we obtain $\mathcal{A}_{\mathcal{F},\lambda} = r \cdot \mathcal{A}_\lambda$; this choice corresponds to the incorrect impression [7] (suggested by a local analysis) that the bc_r-system is just a sum of r usual bc-systems. In the case that we do not assume $\mathcal{F}$ to be of simpler type than the general $\mathcal{E}$ we may set $\lambda = 0$ in (6) and use (3) to recover (5). If we consider families of spin curves [16], i.e., curves with a spin structure (essentially a square root of the canonical bundle), we have to take a finite covering of the moduli space $\mathcal{M}_g$ and are allowed to consider $\lambda \in \frac{1}{2}\mathbb{Z}$. Inserting $\lambda = \frac{1}{2}$ in (6) yields $\mathcal{A}_{\mathcal{F},\frac{1}{2}} = r \cdot \mathcal{A}_{\frac{1}{2}} + \frac{1}{2}\pi_*(c_1^2(\mathcal{F})) - \pi_*(c_2(\mathcal{F}))$. From (3) we inherit the following symmetry around $\frac{1}{2}$:

$$\mathcal{A}_{\mathcal{F},1-\lambda} = \mathcal{A}_{\mathcal{F},\lambda} + (1-2\lambda) \cdot \pi_*(c_1(\omega)c_1(\mathcal{F})).$$

Making this more explicit, we first obtain for $\kappa \in \frac{1}{2}\mathbb{Z}$ that

$$\mathcal{A}_{\mathcal{F},\lambda+\kappa} = \mathcal{A}_{\mathcal{F},\lambda} + \kappa \cdot \pi_*(c_1(\omega)c_1(\mathcal{F})) + 6r\kappa(\kappa + 2\lambda - 1) \cdot \mathcal{A}_1$$

which reduces in the case $\lambda = \frac{1}{2}$ to

$$\mathcal{A}_{\mathcal{F},\frac{1}{2}+\kappa} = \mathcal{A}_{\mathcal{F},\frac{1}{2}} + \kappa \cdot \pi_*(c_1(\omega)c_1(\mathcal{F})) + 6r\kappa^2 \cdot \mathcal{A}_1.$$

This shows explicitly that the term in the middle of the right-hand side destroys the symmetry $\kappa \leftrightarrow -\kappa$, since $\mathcal{A}_{\mathcal{F},\frac{1}{2}+\kappa} - \mathcal{A}_{\mathcal{F},\frac{1}{2}-\kappa} = 2\kappa \cdot \pi_*(c_1(\omega)c_1(\mathcal{F}))$. Note that we do restore this symmetry in the case that $c_1(\mathcal{F}) = 0$. Recall that the symmetry (3) comes from the fact that the *bc*-system is symmetric under interchange of the field contents, i.e., there is no difference in considering b (resp. c) as a section of K^λ (resp. $K^{1-\lambda}$) or $K^{1-\lambda}$ (resp. K^λ). In the bc_r-system we have a symmetry under interchange of E and $K \otimes E^\vee$, see [7]. Since this symmetry should also hold in the relative case, we expect that $\mathcal{A}_{\omega\otimes\mathcal{E}^\vee} = \mathcal{A}_{\mathcal{E}}$. Let us check this explicitly for $\mathcal{E} = \mathcal{F} \otimes \omega^\lambda$, so $\omega \otimes \mathcal{E}^\vee = \mathcal{F}^\vee \otimes \omega^{1-\lambda}$. Using the relation $c_i(\mathcal{F}^\vee) = (-1)^i c_i(\mathcal{F})$ [9], we find with (3) and (6)

$$\begin{aligned}\mathcal{A}_{\mathcal{F}^\vee,1-\lambda} &= r \cdot \mathcal{A}_{1-\lambda} + \frac{1}{2}\pi_*(c_1^2(\mathcal{F}^\vee)) + \frac{2(1-\lambda)-1}{2}\pi_*(c_1(\omega)c_1(\mathcal{F}^\vee)) - \pi_*(c_2(\mathcal{F}^\vee)) \\ &= \mathcal{A}_{\mathcal{F},\lambda}\end{aligned}$$

as we expected. Using the above formulae it is easy to check that in the general case the expected formula holds, i.e.,

$$\mathcal{A}_{\omega\otimes\mathcal{E}^\vee} = \mathcal{A}_{\mathcal{E}}.$$

This is again a consequence of Serre duality and reduces to (3) if one chooses $\mathcal{E} = \omega^\lambda$.

Since the characteristic class $c_2(\mathcal{E})$ appears in formula (5) for the anomaly, it is not possible (in contrast to the usual rank-one case appearing in string theory—recall the introduction) to consider the relative bc_r-system together with a simple bosonic system (like the X^μ) and arrange for a cancellation of anomalies. In particular, one is tempted to introduce an additional system based on $\mathcal{E}^\vee$ to get rid of this term (some kind of 'ghosts of ghosts'), but the symmetry $c_2(\mathcal{E}^\vee) = c_2(\mathcal{E})$ destroys these hopes. Consequently, it will be much more difficult to construct a complete system free of anomalies, but see, e.g., [6].

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